

Large deviation principles for functionals of fractional Brownian motions

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Abstract We study small noise large deviation asymptotics for functionals of fractional Brownian motions. A general sufficient condition for a LDP, formulated in terms of weak convergence properties of certain controlled analogues of the original functionals, is presented. As an application, we prove large deviation principles for a class of stochastic differential equations with a multiplicative noise given as a fractional Brownian motion B^H with Hurst parameter $H > \frac{1}{2}$. The methods presented have broader applicability than the model considered here, for example to systems driven by more general Gaussian noises, and infinite dimensional stochastic dynamical systems with fractional Gaussian noises.

1 Introduction

Theory of large deviations has a long history and several monographs and books have been devoted to the subject [34, 12, 13, 17, 14, 15, 5]. One important topic in this area is the study of asymptotic properties of dynamical systems perturbed by a small noise. When the driving noise is a Brownian motion, such problems have been extensively studied starting from the seminal work of Freidlin and Wentzell [17] (cf. [1], [32], [34], [14]). The original approach for studying such problems relies on techniques based on approximations, discretizations, and exponential probability estimates. These approaches become hard to implement in complex settings, for example for infinite dimensional models, or when the equation coefficients have poor regularity, or when the noise models are non-standard. In recent years (cf. [3, 4, 7, 5]) an alternative approach based on stochastic control ideas and weak convergence methods has emerged, for study of such small noise large deviation problems, that

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has now been applied for a diverse collection of problems (cf. references in [5]). Most of these works consider systems driven by a Brownian noise and/or a Poisson noise (see [8, 6, 5]).

The goal of the current work is to present an approach analogous to the one developed for systems driven by Lévy noises in [4, 7, 8, 5], for dynamical systems driven by a d -dimensional fractional Brownian motion (fBM) B^H , where $H \in (0, 1)$ is the Hurst parameter of the noise process. When $H = 1/2$, $B^H \equiv B$ is a d -dimensional standard Brownian motion and, as noted above, for this case the large deviation properties of small noise stochastic dynamical systems have been well studied. Since for $H \neq 1/2$ the fractional Brownian motion B^H is not a semimartingale, the classical stochastic calculus methods are not readily applicable for the study of stochastic integrals with respect to B^H . Nevertheless, fractional Brownian motion has a nice representation as a Volterra type integral with respect to a standard Brownian motion (cf. [11]):

$$B_t^H = \int_0^1 K_H(t, s) dB_s, \quad t \in [0, 1], \quad (1)$$

where B is a standard Brownian motion and $K_H : [0, 1] \times [0, 1] \rightarrow \mathbb{R}$ is a suitable kernel (see (9)). Several authors have studied stochastic Volterra equations of the form

$$X_t = x_0 + \int_0^t K(t, s)b(s, X_s)ds + \int_0^t K(t, s)\sigma(s, X_s)dB_s, \quad (2)$$

where the above stochastic integral is the usual Itô integral and K is some kernel. Stochastic Volterra equations with regular kernels and driven by a finite-dimensional Brownian motion were first investigated by Berger and Mizel [2] and since then such equations have been widely studied under more general conditions, including settings where K is the fractional Brownian motion kernel K_H (cf. [29], [28], [9], [10], [35]).

Large deviations for small noise stochastic Volterra equations have also been studied in many works including, [27], [23], [38] and [33]. In particular, using Azencott's method (cf. [1]), Nualart and Rovira [27] proved a large deviation principle (LDP) in the space of continuous functions and later Lakhel [23] improved this result by establishing a LDP in a Besov-Orlicz space. As will be seen later, in many important settings stochastic differential equations (SDE) driven by a fBM cannot be represented as a stochastic Volterra equation. It is therefore of interest to develop a general approach, analogous to the one developed in [3, 4] for Brownian motion systems, for studying large deviation results for functionals of a fractional Brownian motion, that cover such SDE models and which does not make an explicit use of the representation (1). In this note we present such an approach that is applicable to a broad range of stochastic dynamical systems driven by fractional Brownian motions. Furthermore, as will be clear from the proof methods, this approach can be extended to other types of Gaussian noises and also for stochastic partial differential equations with fractional Gaussian noises. These extensions will be considered in future works.

The starting point of our work is a nice extension of the variational representation for functionals of Brownian motions given in [3, 4] to the general setting of an abstract Wiener space that was established in [39]. By explicitly identifying the

abstract Wiener space and related constructions for a fractional Brownian motion (e.g. monotonic resolution of the identity operator on the Cameron-Martin space, a filtration on the associated abstract Wiener space, the Skorohod integral) we apply [39, Theorem 3.2] to give a convenient representation for functionals of a fractional Brownian motion that is analogous to Brownian motion representations in [3, 4] (see Proposition 2). Next, by adapting ideas from [3, 4], this representation is used to give a general sufficient condition for a LDP to hold for functionals of a fractional Brownian motion. This condition is analogous to sufficient conditions given for Brownian noise driven systems in [4, 7] and for systems with a Poisson noise in [8] which have found broad applicability (see [5] and references therein). With this sufficient condition, the proofs of LDP for stochastic dynamical systems driven by fBM become very similar to proofs for SDE driven by BM. Once again the key step is to verify certain weak convergence properties of controlled analogues of the original system. The control processes this time take a different form and are given as suitable random non-anticipative elements of the Cameron-Martin space associated with the fractional Brownian motion (see the definition of classes \mathcal{A} and \mathcal{A}_b in Section 3.1). The verification of this sufficient condition is usually the most technical part of the proof of a large deviation principle and depends on model specific properties.

To illustrate the applicability of this method we study large deviation asymptotics for singularly perturbed ordinary differential equations with a multiplicative fractional Brownian noise. Specifically, we consider multidimensional stochastic differential equation (SDE) of the form

$$X_t = x_0 + \int_0^t b(s, X_s) ds + \sqrt{\varepsilon} \int_0^t \sigma(s, X_s) dB_s^H, \quad (3)$$

or equivalently

$$X_t^i = x_0^i + \int_0^t b^i(s, X_s) ds + \sqrt{\varepsilon} \sum_{j=1}^d \int_0^t \sigma^{i,j}(s, X_s) dB_s^{H,j}, \quad i = 1, \dots, m,$$

where $x_0 \in \mathbb{R}^m$, $\varepsilon \in (0, \infty)$ is a small parameter, $b : [0, 1] \times \mathbb{R}^m \rightarrow \mathbb{R}^m$, $\sigma : [0, 1] \times \mathbb{R}^m \rightarrow \mathbb{R}^{m \times d}$ are suitable coefficients, B^H is a d -dimensional fractional Brownian motion with Hurst parameter $H \in (\frac{1}{2}, 1)$, and the integral with respect to B^H is a pathwise Riemann-Stieltjes integral in the sense of [36]. The existence and uniqueness, and the properties of the solution to the above SDE have been studied by many authors (cf. [25], [24], [21], [22], [30], [26], [20]). This SDE cannot in general be written as a stochastic Volterra equation of the form (2). Writing the kernel K_H as $K_H(t, s) = k_H(t, s) \mathbf{1}_{[0, t]}(s)$ where k_H is as in (10) and using the representation (1), one can formally write the pathwise integral $\int_0^t \sigma(s, X_s) dB_s^H$ in (3) as

$$\begin{aligned}
\int_0^t \sigma(s, X_s) dB_s^H &= \int_0^t \sigma(s, X_s) k_H(t, s) dB_s \\
&\quad + \int_0^t \int_s^t [\sigma(u, X_u) - \sigma(s, X_s)] \partial_u k_H(u, s) du \delta B_s \\
&\quad + \int_0^t \int_0^u D_s^B \sigma(u, X_u) \partial_u k_H(u, s) ds du \\
&= \int_0^t \int_s^t \sigma(u, X_u) \partial_u k_H(u, s) \delta B_s \\
&\quad + \int_0^t \int_0^u D_s^B \sigma(u, X_u) \partial_u k_H(u, s) ds du,
\end{aligned}$$

where dB and δB denote the Itô integral and the Skorohod integral respectively with respect to the Brownian motion B , and D^B is the Malliavin derivative with respect to B . In particular, even though B^H has a simple representation in terms of a standard Brownian motion B and the fBm kernel K_H (as given in (1)) the stochastic integral on the left side of the above display does not admit a similar tractable representation. Thus, one cannot directly use the methods of [27, 23, 38, 33] to obtain a large deviation principle for the solution of (3). Our proof proceeds by verifying the general sufficient condition in terms of weak convergence properties of certain controlled analogues of this SDE which is presented in Theorem 1. The verification of these weak convergence properties uses various results for the SDE (3) that have been developed in [26] and a key estimate on the Hölder norms of the solutions in terms of those of the driving paths (see Proposition 3).

The paper is organized as follows. Section 2 introduces some notation and background results. Section 3 gives our main assumptions and states the main results. Proofs of these results are provided in Section 4.

2 Notation and Preliminaries

In this section we recall the definition of a fractional Brownian motion and introduce some related terminology and notation.

2.1 Fractional integrals and derivatives

Let $a, b \in \mathbb{R}$ with $a < b$. Denote by $L^p([a, b])$, $p \geq 1$, the usual space of Lebesgue measurable functions $f : [a, b] \rightarrow \mathbb{R}$ for which $\|f\|_{L^p} < \infty$, where

$$\|f\|_{L^p} = \begin{cases} \left(\int_a^b |f(t)|^p dt \right)^{1/p}, & \text{if } 1 \leq p < \infty \\ \text{ess sup}\{|f(t)| : t \in [a, b]\}, & \text{if } p = \infty. \end{cases}$$

Let $f \in L^1([a, b])$ and $\alpha > 0$. The left-sided and right-sided fractional Riemann-Liouville integrals of f of order α are defined for almost all $t \in (a, b)$ by

$$I_{a+}^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} f(s) ds$$

and

$$I_{b-}^\alpha f(t) = \frac{(-1)^{-\alpha}}{\Gamma(\alpha)} \int_t^b (s-t)^{\alpha-1} f(s) ds,$$

respectively, where $(-1)^{-\alpha} = e^{-i\pi\alpha}$ and $\Gamma(\alpha) = \int_0^\infty r^{\alpha-1} e^{-r} dr$ is the Euler gamma function. Let $I_{a+}^\alpha(L^p([a, b]))$ (resp. $I_{b-}^\alpha(L^p([a, b]))$) be the image of $L^p([a, b])$ under the operator I_{a+}^α (resp. I_{b-}^α). If $f \in I_{a+}^\alpha(L^p([a, b]))$ (resp. $f \in I_{b-}^\alpha(L^p([a, b]))$) and $0 < \alpha < 1$ then the Weyl derivatives are defined as

$$D_{a+}^\alpha f(t) = \frac{1}{\Gamma(1-\alpha)} \left(\frac{f(t)}{(t-a)^\alpha} + \alpha \int_a^t \frac{f(t) - f(s)}{(t-s)^{\alpha+1}} ds \right) 1_{(a,b)}(t) \quad (4)$$

and

$$D_{b-}^\alpha f(t) = \frac{(-1)^\alpha}{\Gamma(1-\alpha)} \left(\frac{f(t)}{(b-t)^\alpha} + \alpha \int_t^b \frac{f(t) - f(s)}{(s-t)^{\alpha+1}} ds \right) 1_{(a,b)}(t) \quad (5)$$

for almost all $t \in (a, b)$ (the convergence of the integrals at the singularity $s = t$ holds pointwise for almost all $t \in (a, b)$ if $p = 1$ and in L^p if $1 < p < \infty$).

Denote by $C([a, b] : \mathbb{R}^d)$ the space of \mathbb{R}^d -valued continuous functions on the interval $[a, b]$. Note that $C([a, b] : \mathbb{R}^d)$ is a separable Banach space equipped with the following norm

$$\|f\|_{a,b,\infty} = \sup_{a \leq r \leq b} |f(r)|.$$

For any $\lambda \in (0, 1)$, we denote by $C^\lambda([a, b] : \mathbb{R}^d)$ the space of \mathbb{R}^d -valued λ -Hölder continuous functions on the interval $[a, b]$. We will make use of the notation

$$\|f\|_{a,b,\lambda} = \sup_{a \leq \theta < r \leq b} \frac{|f(r) - f(\theta)|}{|r - \theta|^\lambda},$$

if $f \in C^\lambda([a, b] : \mathbb{R}^d)$. In the case of $d = 1$, we abbreviate $C^\lambda([a, b] : \mathbb{R})$ as $C^\lambda([a, b])$.

Suppose that $f \in C^\lambda([a, b])$ and $g \in C^\mu([a, b])$ with $\lambda + \mu > 1$. Then, from the classical paper by Young [36], the Riemann-Stieltjes integral $\int_a^b f dg$ exists. The following proposition can be regarded as a fractional integration by parts formula, and provides an explicit expression for the integral $\int_a^b f dg$ in terms of fractional derivatives (cf. [37]).

Proposition 1 *Suppose that $f \in C^\lambda([a, b])$ and $g \in C^\mu([a, b])$ with $\lambda + \mu > 1$. Let $0 < \alpha < 1$, $\lambda > \alpha$ and $\mu > 1 - \alpha$. Then the Riemann-Stieltjes integral $\int_a^b f dg$ exists*

and it can be expressed as

$$\int_a^b f dg = (-1)^\alpha \int_a^b D_{a+}^\alpha f(t) D_{b-}^{1-\alpha} g_{b-}(t) dt, \quad (6)$$

where $g_{b-}(t) = g(t) - g(b)$.

When we consider the interval $[a, b] = [0, 1]$, we will use the following simplified notations

$$\|f\|_\infty = \|f\|_{0,1,\infty}, \quad \|f\|_\lambda = \|f\|_{0,1,\lambda}$$

for f in $C([0, 1] : \mathbb{R}^d)$ and $C^\lambda([0, 1] : \mathbb{R}^d)$, respectively.

For $0 < \alpha < 1$, let $W_0^{\alpha,\infty}([0, 1] : \mathbb{R}^d)$ be the space of \mathbb{R}^d -valued measurable functions $f : [0, 1] \rightarrow \mathbb{R}^d$ such that

$$\|f\|_{\alpha,\infty} = \sup_{0 \leq t \leq 1} \left\{ |f(t)| + \int_0^t \frac{|f(t) - f(s)|}{(t-s)^{\alpha+1}} ds \right\} < \infty.$$

It is easy to see that

$$C^{\alpha+\varepsilon}([0, 1] : \mathbb{R}^d) \subseteq W_0^{\alpha,\infty}([0, 1] : \mathbb{R}^d)$$

for any $\varepsilon > 0$. On the other hand, by the Garsia-Rademich-Rumsey inequality (see [18]), it follows that

$$W_0^{\alpha,\infty}([0, 1] : \mathbb{R}^d) \subseteq C^{\alpha-\varepsilon}([0, 1] : \mathbb{R}^d)$$

for any $0 < \varepsilon < \alpha$.

2.2 Fractional Brownian motion

We now recall the definition of a fractional Brownian motion.

Definition 1 For $H \in (0, 1)$, a d -dimensional fractional Brownian motion (fBm for short) $B^H = \{B_t^H : t \in [0, 1]\}$ with Hurst parameter H defined on a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$ is a centered Gaussian process whose covariance matrix $R_H = (R_H^{i,j})_{1 \leq i, j \leq d}$ is given by

$$R_H^{i,j}(s, t) = \mathbb{E}(B_s^{H,i} B_t^{H,j}) = \frac{1}{2}(s^{2H} + t^{2H} - |t - s|^{2H})\delta_{i,j}, \quad s, t \in [0, 1], \quad (7)$$

where δ is the Kronecker delta function.

For $H = \frac{1}{2}$, the process $B^{\frac{1}{2}}$ is a d -dimensional standard Brownian motion. For $H \neq \frac{1}{2}$, the fBm B^H is not a semimartingale. It follows from (7) that

$$\mathbb{E}(|B_t^H - B_s^H|^2) = d|t - s|^{2H}, \quad t, s \in [0, 1].$$

From the above property along with Kolmogorov's continuity criterion it follows that the sample paths of B^H are a.s. Hölder continuous of order β for all $\beta < H$.

In the sequel we consider the canonical probability space $(\Omega, \mathcal{F}, \mathbb{P})$, where $\Omega = C_0([0, 1] : \mathbb{R}^d)$ is the space of continuous functions null at time 0, $\mathcal{F} = \mathcal{B}(C_0([0, 1] : \mathbb{R}^d))$ is the Borel σ -algebra and \mathbb{P} is the unique d -dimensional probability measure such that the canonical process $B^H = \{B_t^H(\omega) = \omega(t) : t \in [0, 1]\}$ is a d -dimensional fractional Brownian motion with Hurst parameter H . Consider the canonical filtration given by $\{\mathcal{F}_t^H : t \in [0, 1]\}$, where $\mathcal{F}_t^H = \sigma\{B_s^H : 0 \leq s \leq t\} \vee \mathcal{N}$ and \mathcal{N} is the set of the \mathbb{P} -negligible events.

Let $F(a, b, c; z)$ denote the Gauss hypergeometric function defined for any $a, b, c, z \in \mathbb{C}$ with $|z| < 1$ and $c \neq 0, -1, -2, \dots$ by

$$F(a, b, c; z) = \sum_{k=0}^{\infty} \frac{(a)_k (b)_k}{(c)_k} z^k,$$

where $(a)_0 = 1$ and $(a)_k = a(a+1) \dots (a+k-1)$ is the Pochhammer symbol.

Let $B = \{B_t = (B_t^1, \dots, B_t^d), t \in [0, 1]\}$ be a standard d -dimensional Brownian motion. Then from [11] we have that the process

$$B_t^H = \int_0^1 K_H(t, s) dB_s, \quad t \in [0, 1] \quad (8)$$

defines a fBm with Hurst parameter H , where

$$K_H(t, s) = k_H(t, s) \mathbf{1}_{[0, t]}(s), \quad (9)$$

for $0 \leq s \leq t$

$$k_H(t, s) = \frac{c_H}{\Gamma\left(H + \frac{1}{2}\right)} (t-s)^{H-\frac{1}{2}} F\left(H - \frac{1}{2}, \frac{1}{2} - H, H + \frac{1}{2}; 1 - \frac{t}{s}\right), \quad (10)$$

$$c_H = \left[\frac{2H\Gamma(\frac{3}{2}-H)\Gamma(H+\frac{1}{2})}{\Gamma(2-2H)} \right]^{1/2}, \text{ and } \Gamma(\cdot) \text{ as before is the gamma function.}$$

The next lemma (cf. Theorem 2.1 in [11] and (10.22) in [31]) will play an important role in the construction of the Cameron-Martin space in the abstract Wiener space associated with a fBm.

Lemma 1 For $H \in (0, 1)$, consider the integral transform

$$\begin{aligned}
(K_H f)(t) &= \int_0^1 K_H(t, s) f(s) ds \\
&= \frac{c_H}{\Gamma(H + \frac{1}{2})} \int_0^1 (t-s)^{H-\frac{1}{2}} F\left(\frac{1}{2} - H, H - \frac{1}{2}, H + \frac{1}{2}; 1 - \frac{t}{s}\right) f(s) ds. \quad (12)
\end{aligned}$$

Then K_H is an isomorphism from $L^2([0, 1] : \mathbb{R}^d)$ onto $I_{0+}^{H+\frac{1}{2}}(L^2([0, 1] : \mathbb{R}^d))$ and

$$K_H f(t) = c_H I_{0+}^1 \left(\psi \left(I_{0+}^{H-\frac{1}{2}} (\psi^{-1} f) \right) \right) (t), \text{ if } H \geq \frac{1}{2}, t \in [0, 1] \quad (13)$$

$$K_H f(t) = c_H I_{0+}^{2H} \left(\psi^{-1} \left(I_{0+}^{\frac{1}{2}-H} (\psi f) \right) \right) (t), \text{ if } H \leq \frac{1}{2}, t \in [0, 1]. \quad (14)$$

where $\psi(u) = u^{H-\frac{1}{2}}$ for $u \in [0, 1]$.

Remark 1 For any $a < b$, if $\alpha > \frac{1}{p}$, then from Theorem 3.6 in [31] we have

$$I_{a+}^{\alpha}(L^p([a, b])) \cup I_{b-}^{\alpha}(L^p([a, b])) \subset C^{\alpha-\frac{1}{p}}([a, b]).$$

In particular, we have $I_{0+}^{H+\frac{1}{2}}(L^2([0, 1] : \mathbb{R}^d)) \subset C^H([0, 1] : \mathbb{R}^d)$. Note that $(K_H f)(0) = 0$ for all $f \in L^2([0, 1] : \mathbb{R}^d)$. Thus K_H maps $L^2([0, 1] : \mathbb{R}^d)$ into $C_0([0, 1] : \mathbb{R}^d)$ and for $f_1, f_2 \in L^2([0, 1] : \mathbb{R}^d)$, $f_1 = f_2$ a.e. if and only if $K_H f_1 = K_H f_2$.

Define $\mathcal{H}_H = \{(K_H \dot{h}^1, \dots, K_H \dot{h}^d) : \dot{h} = (\dot{h}^1, \dots, \dot{h}^d) \in L^2([0, 1] : \mathbb{R}^d)\}$, that is, any $h \in \mathcal{H}_H$ can be represented as

$$h(t) = (K_H \dot{h})(t) = \int_0^1 K_H(t, s) \dot{h}(s) ds,$$

for some $\dot{h} \in L^2([0, 1] : \mathbb{R}^d)$. Define a scalar inner product on \mathcal{H}_H by

$$\langle h, g \rangle_{\mathcal{H}_H} = \langle K_H \dot{h}, K_H \dot{g} \rangle_{\mathcal{H}_H} = \langle \dot{h}, \dot{g} \rangle_{L^2}.$$

Then \mathcal{H}_H is a separable Hilbert space with the inner product $\langle \cdot, \cdot \rangle_{\mathcal{H}_H}$. From Remark 1, we see that \mathcal{H}_H is a subset of $\Omega = C_0([0, 1] : \mathbb{R}^d)$.

3 Results

We now present the main results of this work. The proofs will be provided in Section 4.

3.1 A variational representation for functionals of fBm

For $0 < N < \infty$, let $S_N = \{v \in \mathcal{H}_H : \frac{1}{2}\|v\|_{\mathcal{H}_H}^2 \leq N\}$. Equipped with the weak topology on \mathcal{H}_H , S_N can be metrized as a compact Polish space. Let \mathcal{A} denote the class of all \mathcal{H}_H -valued random variables v in $L^2((\Omega, \mathcal{F}, \mathbb{P}) : \mathcal{H}_H)$ such that $v(t)$ is $\{\mathcal{F}_t^H\}$ -measurable for every $t \in [0, 1]$. The set of all a.s. bounded elements in \mathcal{A} is denoted by \mathcal{A}_b , that is,

$$\mathcal{A}_b = \{v \in \mathcal{A} : \|v(\omega)\|_{\mathcal{H}_H} \leq N \text{ } \mathbb{P}\text{-a.s. for some } 0 < N < \infty\}.$$

The following variational representation is obtained by making use of [39, Theorem 3.2]. The proof is provided in Section 4.

Proposition 2 *Let f be a bounded Borel measurable function on Ω . Then we have*

$$\begin{aligned} -\log \mathbb{E}(e^{-f(\omega)}) &= \inf_{v \in \mathcal{A}} \mathbb{E} \left(f(\omega + v) + \frac{1}{2} \|v\|_{\mathcal{H}_H}^2 \right) \\ &= \inf_{v \in \mathcal{A}_b} \mathbb{E} \left(f(\omega + v) + \frac{1}{2} \|v\|_{\mathcal{H}_H}^2 \right). \end{aligned}$$

3.2 A general large deviation principle

We begin by recalling the definition of a large deviation principle. Let \mathcal{E} be a Polish space (a complete separable metric space) and let $\{X^\varepsilon : \varepsilon \in (0, 1)\}$ be a collection of \mathcal{E} -valued random variables.

Definition 2 (a) A function $I : \mathcal{E} \rightarrow [0, \infty]$ is called a rate function on \mathcal{E} , if for each $M < \infty$ the level set $\{x \in \mathcal{E} : I(x) \leq M\}$ is a compact subset of \mathcal{E} . For $A \in \mathcal{B}(\mathcal{E})$, we define $I(A) = \inf_{x \in A} I(x)$.

(b) Let I be a rate function on \mathcal{E} . A collection $\{X^\varepsilon : \varepsilon \in (0, 1)\}$ of \mathcal{E} -valued random variables is said to satisfy a large deviation principle in \mathcal{E} , as $\varepsilon \rightarrow 0$, with rate function I if the following two conditions hold:

(i) Large deviation upper bound. For each open set G in \mathcal{E} ,

$$\limsup_{\varepsilon \rightarrow 0} -\varepsilon \log \mathbb{P}(X^\varepsilon \in G) \leq I(G).$$

(ii) Large deviation lower bound. For each closed set F in \mathcal{E} ,

$$\liminf_{\varepsilon \rightarrow 0} -\varepsilon \log \mathbb{P}(X^\varepsilon \in F) \geq I(F).$$

For $0 < \varepsilon < 1$ let $\mathcal{G}^\varepsilon : \Omega = C_0([0, 1] : \mathbb{R}^d) \rightarrow \mathcal{E}$ be a measurable map. We will now give a sufficient condition on the maps \mathcal{G}^ε for a LDP to hold for the collection of \mathcal{E} -valued random variables

$$X^\varepsilon(\omega) = \mathcal{G}^\varepsilon(\sqrt{\varepsilon}\omega), \quad \omega \in \Omega. \quad (15)$$

Recall that S_N is a compact metric space that is equipped with the weak topology inherited from \mathcal{H}_H . Also recall the canonical filtration $\{\mathcal{F}_t^H\}$ on $(\Omega, \mathcal{F}, \mathbb{P})$ and the classes $\mathcal{A}, \mathcal{A}_b$.

Assumption There exists a measurable map $\mathcal{G}^0 : \mathcal{H}_H \rightarrow \mathcal{E}$ such that the following conditions hold.

- (i) Consider $0 < N < \infty$ and a family of S_N -valued random elements $\{v^\varepsilon\} \subset \mathcal{A}_b$ on $(\Omega, \mathcal{F}, \mathbb{P})$ such that v^ε converges in distribution to v . Then $\mathcal{G}^\varepsilon(\sqrt{\varepsilon}\omega + v^\varepsilon)$ converges to $\mathcal{G}^0(v)$ in distribution.
- (ii) For every $0 < N < \infty$, the set

$$\Gamma_N = \{\mathcal{G}^0(v) : v \in S_N\}$$

is a compact subset of \mathcal{E} . □

For $x \in \mathcal{E}$, define

$$I(x) = \inf_{\{v \in \mathcal{H}_H : x = \mathcal{G}^0(v)\}} \left\{ \frac{1}{2} \|v\|_{\mathcal{H}_H}^2 \right\} \quad (16)$$

whenever $\{v \in \mathcal{H}_H : x = \mathcal{G}^0(v)\} \neq \emptyset$, and $I(x) = \infty$ otherwise.

The following theorem gives a LDP under Assumption 1. Proof is given in Section 4.

Theorem 1 Let X^ε be as defined in (15). Suppose that $\{\mathcal{G}^\varepsilon\}$ satisfies Assumption 1. Then the family $\{X^\varepsilon : \varepsilon \in (0, 1)\}$ satisfies a large deviation principle in \mathcal{E} , as $\varepsilon \rightarrow 0$, with rate function I defined in (16).

3.3 SDE with a multiplicative fractional noise

For $\varepsilon > 0$, consider the following stochastic differential equation

$$X_t^\varepsilon = x_0 + \int_0^t b(s, X_s^\varepsilon) ds + \sqrt{\varepsilon} \int_0^t \sigma(s, X_s^\varepsilon) dB_s^H, \quad t \in [0, 1]. \quad (17)$$

where B^H is a d -dimensional fBm with $H > 1/2$. For wellposedness of the above equation we introduce the following condition on the coefficients b and σ . For a matrix $A = (a^{i,j})_{m \times d}$ and a vector $y = (y^i)_{m \times 1}$, denote $|A|^2 = \sum_{i,j} |a^{i,j}|^2$ and $|y|^2 = \sum_i |y^i|^2$.

Assumption (i) The function $\sigma : [0, 1] \times \mathbb{R}^m \rightarrow \mathbb{R}^{m \times d}$ is differentiable in the space variable $x \in \mathbb{R}^m$, and there exist constants $M > 0$, $1 - H < \lambda \leq 1$, $\frac{1}{H} - 1 < \gamma \leq 1$, and for every $N > 0$ there exists $M_N > 0$, such that the following hold:

$$|\sigma(t, x) - \sigma(t, y)| \leq M|x - y|, \text{ for all } (t, x) \in [0, 1] \times \mathbb{R}^m,$$

$$|\partial_{x_i} \sigma(t, x) - \partial_{y_i} \sigma(t, y)| \leq M_N |x - y|^\gamma, \text{ for all } t \in [0, 1], x, y \in \mathbb{R}^m, |x| \vee |y| \leq N,$$

$$|\sigma(t, x) - \sigma(s, x)| + |\partial_{x_i} \sigma(t, x) - \partial_{x_i} \sigma(s, x)| \leq M|t - s|^\lambda, \text{ for all } x \in \mathbb{R}^m, t, s \in [0, 1],$$

for each $i = 1, \dots, m$.

(ii) There exists $L > 0$ such that the following properties are satisfied:

$$|b(t, x) - b(t, y)| \leq L|x - y|, \quad |b(t, x)| \leq L(1 + |x|) \text{ for all } x, y \in \mathbb{R}^m \text{ and } t \in [0, 1].$$

Remark 2 From Assumption 2(i) it follows that there exists a $K > 0$ such that

$$|\sigma(t, x)| \leq K(1 + |x|), \text{ for all } (t, x) \in [0, 1] \times \mathbb{R}^m. \quad (18)$$

Let $g : [0, 1] \rightarrow \mathbb{R}^d$ be such that for any $0 < \varepsilon < H$, $g \in C^{H-\varepsilon}([0, 1] : \mathbb{R}^d)$. Consider the deterministic equation on \mathbb{R}^m

$$x_t = x_0 + \int_0^t b(s, x_s) ds + \int_0^t \sigma(s, x_s) dg_s, \quad (19)$$

where $x_0 \in \mathbb{R}^m$ and the last integral is interpreted as a Riemann-Stieltjes integral.

The wellposedness in part (a) of the next proposition is taken from [26, Theorem 5.1]. The estimate given in part (b) of the proposition will play an important role in the analysis and will be established in Section 4.3.

Proposition 3 *Suppose that the coefficients b and σ satisfy Assumption 2. Then the following hold.*

(a) *Equation (19) has a unique solution $x \in W_0^{\alpha, \infty}([0, 1] : \mathbb{R}^m)$ for any $\alpha \in (1 - H, \min\{\frac{1}{2}, \lambda, \frac{\gamma}{1+\gamma}\})$. Moreover, the solution is $(1 - \alpha)$ -Hölder continuous.*

(b) *For any $\alpha \in (1 - H, \min\{\frac{1}{2}, \lambda, \frac{\gamma}{1+\gamma}\})$ and any $0 < \delta < \alpha - (1 - H)$, the following estimates hold:*

$$\|x\|_\infty \leq C_1(1 + |x_0|) \exp\{C_2 \|g\|_{1-\alpha+\delta}^\kappa\}, \quad (20)$$

$$\|x\|_{1-\alpha} \leq C_3(1 + |x_0|)(1 + \|g\|_{1-\alpha+\delta}^\kappa)(1 + \|g\|_{1-\alpha+\delta})(1 + \exp\{C_2 \|g\|_{1-\alpha+\delta}^\kappa\})$$

where $\kappa = \frac{1}{1-\alpha}$ and the constants C_1 , C_2 and C_3 depend on α , δ and all the constants appearing in Assumption 2.

Remark 3 As observed above Proposition 1, for $f \in C^{\mu_1}([a, b] : \mathbb{R})$ and $h \in C^{\mu_2}([a, b] : \mathbb{R})$ with $\mu_1 + \mu_2 > 1$, the Riemann-Stieltjes integral $\int_a^b f_s dh_s$ exists. From Assumption 2 and the observation that $W_0^{\alpha, \infty}([0, 1] : \mathbb{R}^m) \subseteq C^{\alpha-\varepsilon}([0, 1] : \mathbb{R}^m)$ for any $0 < \varepsilon < \alpha$, it follows that if for some $\alpha > 1 - H$, $x \in W_0^{\alpha, \infty}([0, 1] : \mathbb{R}^m)$ then $s \mapsto \sigma(s, x(s)) \in C^\delta([0, 1] : \mathbb{R}^m)$ for some $\delta > 1 - H$. Thus, from properties

of g it follows that the integral $\int_0^t \sigma(s, x_s) dg_s$ is well defined as a Riemann-Stieltjes integral.

In Lemma 2 it will be seen that, $\mathcal{H}_H \subset C^H([0, 1] : \mathbb{R}^d)$ and thus from Proposition 3, under Assumption 2, equation (19) with g replaced by v has a unique solution. For $f \in C([0, 1] : \mathbb{R}^m)$, define

$$C_f \doteq \left\{ v \in \mathcal{H}_H : f_t = x_0 + \int_0^t b(s, f_s) ds + \int_0^t \sigma(s, f_s) dv_s \text{ for all } t \in [0, 1] \right\}$$

and let

$$I(f) = \inf_{v \in C_f} \frac{1}{2} \|v\|_{\mathcal{H}_H}^2 \quad (22)$$

whenever $C_f \neq \emptyset$, and $I(f) = \infty$ otherwise.

We now return to SDE (17). Recall that the sample paths of B^H are a.s. in $C^{H-\varepsilon}([0, 1] : \mathbb{R}^d)$ for any $0 < \varepsilon < H$. Thus as an immediate consequence of Proposition 3 and Fernique's theorem (cf. [16]) we have the following result.

Proposition 4 *Let Assumption 2 be satisfied.*

- (a) *For any $\alpha \in (1 - H, \min\{\frac{1}{2}, \lambda, \frac{\gamma}{1+\gamma}\})$, the SDE (17) has a unique pathwise solution X^ε with $X^\varepsilon \in W_0^{\alpha, \infty}([0, 1] : \mathbb{R}^m)$ a.s. Moreover, for \mathbb{P} -almost all $\omega \in \Omega$, the solution $X^\varepsilon(\omega)$ is $(1 - \alpha)$ -Hölder continuous.*
- (b) *The solution X^ε to the SDE (17) satisfies*

$$\sup_{\varepsilon \in (0, 1)} \mathbb{E} \|X^\varepsilon\|_{1-\alpha}^p < \infty, \text{ for all } p \geq 1. \quad (23)$$

The following result gives a LDP for the solution process $\{X^\varepsilon\}$. Proof is in Section 4.

Theorem 2 *Suppose that Assumption 2 is satisfied and let for $\varepsilon \in (0, 1)$, X^ε be defined as in (17). Then the family $\{X^\varepsilon : \varepsilon \in (0, 1)\}$ satisfies a large deviation principle in $C([0, 1] : \mathbb{R}^m)$ with rate function I defined in (22).*

4 Proofs

In this section we give the proofs of Proposition 2, Theorem 1, Proposition 3(b) and Theorem 2.

4.1 Proof of Proposition 2

In this subsection we describe the abstract Wiener space for B^H and, using [39, Theorem 3.2], provide the proof of the variational representation for functionals of fractional Brownian motion given in Proposition 2.

The following elementary lemma shows that functions in \mathcal{H}_H are H -Hölder continuous.

Lemma 2 *Let $h \in \mathcal{H}_H$. Then $h \in C^H([0, 1] : \mathbb{R}^d)$, and $\|h\|_\infty \leq \|h\|_{\mathcal{H}_H}$ and $\|h\|_H \leq \|h\|_{\mathcal{H}_H}$.*

Proof Without loss of generality, we assume $d = 1$. For $h = K_H \dot{h} \in \mathcal{H}_H$ and $t \in [0, 1]$, we have, from the integral representation (8),

$$\begin{aligned} h(t) &= \int_0^1 K_H(t, u) \dot{h}(u) du \\ &= \mathbb{E} \left(\int_0^1 K_H(t, u) dB_u \int_0^1 \dot{h}(u) dB_u \right) = \mathbb{E} \left(B_t^H \int_0^1 \dot{h}(u) dB_u \right), \end{aligned}$$

where B is a standard Brownian motion and consequently $B_t^H = \int_0^t K_H(t, u) dB_u$ is a fractional Brownian motion with Hurst parameter H . Then, for any $s, t \in [0, 1]$, we have

$$|h(t)| \leq \left(\mathbb{E} \left(|B_t^H|^2 \right) \right)^{1/2} \|\dot{h}\|_{L^2} = t^H \|h\|_{\mathcal{H}_H} \leq \|h\|_{\mathcal{H}_H}, \quad (24)$$

and

$$|h(t) - h(s)| \leq \left(\mathbb{E} \left(|B_t^H - B_s^H|^2 \right) \right)^{1/2} \|\dot{h}\|_{L^2} = |t - s|^H \|\dot{h}\|_{L^2} = |t - s|^H \|h\|_{\mathcal{H}_H},$$

which completes the proof. \square

Recall that $\Omega = C_0([0, 1] : \mathbb{R}^d)$ is a Banach space equipped with the sup-norm $\|\cdot\|_\infty$. Let Ω^* be its topological dual.

We now introduce the abstract Wiener space associated with a fractional Brownian motion. The following result is taken from [11, Theorem 3.3].

Lemma 3 *If we identify $L^2([0, 1] : \mathbb{R}^d)$ and its dual, we have the following diagram*

$$\Omega^* \xrightarrow{i_H^*} \mathcal{H}_H^* \xrightarrow{K_H^*} L^2([0, 1] : \mathbb{R}^d) \xrightarrow{K_H} \mathcal{H}_H \xrightarrow{i_H} \Omega$$

Where K_H is defined by (11), i_H is the injection from \mathcal{H}_H into Ω , and K_H^* and i_H^* are the respective adjoints.

(a) The injection i_H embeds \mathcal{H}_H densely into Ω , and \mathcal{H}_H is the Cameron-Martin space of the abstract Wiener space $(i_H, \mathcal{H}_H, \Omega)$ in the sense of Gross [19].

(b) The restriction of K_H^* to Ω^* can be represented by

$$(K_H^* \eta)(s) = \int_0^1 K_H(t, s) \eta(dt) = \left(\int_0^1 K_H(t, s) \eta_1(dt), \dots, \int_0^1 K_H(t, s) \eta_d(dt) \right),$$

for any $\eta = (\eta_1, \dots, \eta_d) \in \Omega^*$.

Next, we introduce the Skorohod integral $\delta(h)$ for $h \in \mathcal{H}_H$. This is well known (cf. [11], [39] and references therein) but we give a self contained presentation for reader's convenience. The injection i_H^* embeds Ω^* densely into \mathcal{H}_H^* , since Ω^* separates \mathcal{H}_H . Define $R_H = K_H \circ K_H^*$, then R_H embeds Ω^* densely into \mathcal{H}_H , namely, for any $h \in \mathcal{H}_H$, there exists a sequence of $\{\eta_n\}_{n=1}^\infty \subset \Omega^*$ such that $\lim_{n \rightarrow \infty} \|R_H \eta_n - h\|_{\mathcal{H}_H} = 0$. Next note that

$$\begin{aligned} & \int |\eta_n(\omega) - \eta_k(\omega)|^2 \mathbb{P}(d\omega) \\ &= \mathbb{E}(|\eta_n(\cdot) - \eta_k(\cdot)|^2) \\ &= \mathbb{E} \left(\sum_{1 \leq i, j \leq d} \int_0^1 \int_0^1 \omega^i(t) \omega^j(s) (\eta_n^i - \eta_k^i)(dt) (\eta_n^j - \eta_k^j)(ds) \right) \\ &= \sum_{1 \leq i, j \leq d} \int_0^1 \int_0^1 R_H^{i,j}(t, s) (\eta_n^i - \eta_k^i)(dt) (\eta_n^j - \eta_k^j)(ds) \\ &= \sum_{1 \leq i, j \leq d} \int_0^1 \int_0^1 \int_0^1 \delta_{i,j} K_H(t, r) K_H(s, r) dr (\eta_n^i - \eta_k^i)(dt) (\eta_n^j - \eta_k^j)(ds) \\ &= \int_0^1 |K_H^*(\eta_n - \eta_k)(r)|^2 dr = \|R_H(\eta_n - \eta_k)\|_{\mathcal{H}_H}^2 \rightarrow 0, \end{aligned}$$

as $n, k \rightarrow \infty$. Therefore, there exists $\delta(h) \in L^2(\Omega, \mathcal{F}, \mathbb{P})$ such that $\lim_{n \rightarrow \infty} \mathbb{E}(|\eta_n(\cdot) - \delta(h)|^2) = 0$. The limit $\delta(h)$ is called the Skorohod integral of h . Moreover, the following isometry property holds

$$\mathbb{E}(\delta(h)\delta(g)) = \langle h, g \rangle_{\mathcal{H}_H}, \text{ for any } h, g \in \mathcal{H}_H. \quad (25)$$

Recall the filtered probability space $(\Omega, \mathcal{F}, \mathbb{P}, \{\mathcal{F}_t^H\})$ introduced below Definition 1 and recall that $\{B_t^H\}$ is the canonical coordinate process on (Ω, \mathcal{F}) .

Define the family $\{\pi_t^H, t \in [0, 1]\}$ of orthogonal projections in \mathcal{H}_H by

$$\pi_t^H h = \pi_t^H (K_H \dot{h}) = K_H(\dot{h} \mathbf{1}_{[0, t]}), \quad h \in \mathcal{H}_H. \quad (26)$$

Define for $i = 1, \dots, d$, $e_i : [0, 1] \rightarrow \mathbb{R}^d$ as

$$(e_i(t))_j = \begin{cases} 0 & \text{if } j \neq i \\ 1 & \text{if } j = i \end{cases}, \quad t \in [0, 1].$$

Then $e_i \in L^2([0, 1] : \mathbb{R}^d)$ for $i = 1, \dots, d$. Define the process $B = (B^1, \dots, B^d)$ by

$$B_t^i = \delta(\pi_t^H K_H e_i), \quad t \in [0, 1], \quad i = 1, \dots, d. \quad (27)$$

If $h \in \mathcal{H}_H$ then, it can be shown that, g defined as

$$g(t) = c_H^{-1} t^{H-\frac{1}{2}} (D_{0+}^{H-\frac{1}{2}} (\psi^{-1} h'))(t), \quad t \in [0, 1]$$

is in $L^2([0, 1] : \mathbb{R}^d)$ and we have $h = K_H g$. Henceforth, for $h \in \mathcal{H}_H$, we will take the function g defined as above the definition of \dot{h} in the representation $h = K_H \dot{h}$.

The following result is a consequence of [11, Proposition 4.4, Theorems 4.3 and 4.8].

Lemma 4 (a) For any $t \in [0, 1]$, $\mathcal{F}_t^H = \sigma\{\delta(\pi_t^H h), h \in \mathcal{H}_H\} \vee \mathcal{N}$.
 (b) A \mathcal{H}_H -valued stochastic process $u = K_H \dot{u}$ is $\{\mathcal{F}_t^H, t \in [0, 1]\}$ -adapted if and only if the process \dot{u} is $\{\mathcal{F}_t^H, t \in [0, 1]\}$ -progressively measurable.
 (c) The process $B = \{B_t = (B_t^1, \dots, B_t^d), t \in [0, 1]\}$ is a \mathbb{P} -standard Brownian motion and $\sigma\{B_s, s \leq t\} \vee \mathcal{N}$ equals \mathcal{F}_t^H for $t \in [0, 1]$.

Proof of Proposition 2. The collection $\{\pi_t^H, t \in [0, 1]\}$ introduced in (26) defines a continuous and strictly monotonic resolution of the identity in \mathcal{H}_H in the sense of [39, Section 2] and furthermore the operator $\delta(\cdot)$ introduced in the latter paper coincides with the Skorohod integral defined above (25). Also, the classes \mathcal{A} and \mathcal{A}_b in Section 3.1 are the same as the classes \mathcal{H}^a and \mathcal{H}_b^a in [39] when specialized to the setting considered here. The result is now immediate from [39, Theorem 3.2].
 \square

4.2 Proof of Theorem 1

The proof follows along the lines of [4, Theorem 4.4] however we provide details for reader's convenience. It will be convenient to work with the following equivalent formulation of a LDP.

Definition 3 Let I be a rate function on some Polish space \mathcal{E} . A collection $\{X^\varepsilon : \varepsilon \in [0, 1]\}$ of \mathcal{E} -valued random variables is said to satisfy the Laplace principle upper bound (lower bound, respectively) on \mathcal{E} with rate function I if for all bounded continuous functions $h : \mathcal{E} \rightarrow \mathbb{R}$,

$$\limsup_{\varepsilon \rightarrow 0} -\varepsilon \log \mathbb{E} \left(\exp \left\{ -\frac{h(X^\varepsilon)}{\varepsilon} \right\} \right) \leq \inf_{x \in \mathcal{E}} \{h(x) + I(x)\}, \quad (28)$$

(respectively,

$$\liminf_{\varepsilon \rightarrow 0} -\varepsilon \log \mathbb{E} \left(\exp \left\{ -\frac{h(X^\varepsilon)}{\varepsilon} \right\} \right) \geq \inf_{x \in \mathcal{E}} \{h(x) + I(x)\}.) \quad (29)$$

The Laplace principle (LP) is said to hold for $\{X^\varepsilon : \varepsilon \in [0, 1]\}$ with rate function I if both the Laplace upper bound and lower bound are satisfied for all bounded continuous functions $h : \mathcal{E} \rightarrow \mathbb{R}$.

It is well known that a collection $\{X^\varepsilon : \varepsilon \in [0, 1]\}$ of \mathcal{E} -valued random variables satisfies a LDP with rate function I if and only if it satisfies a LP with rate function I (cf. [5, Theorems 1.5 and 1.8]).

We now proceed to the proof of Theorem 1.

Proof (Proof of Theorem 1) It suffices to prove (28) and (29), with I as defined in (16), for all real-valued, bounded and continuous functions h on \mathcal{E} , and to prove that I is a rate function.

Proof of the upper bound (28): Without loss we assume that $\inf_{x \in \mathcal{E}} \{h(x) + I(x)\} < \infty$. Let $\delta > 0$ be arbitrary. Then there exists $x_0 \in \mathcal{E}$ such that

$$h(x_0) + I(x_0) \leq \inf_{x \in \mathcal{E}} \{h(x) + I(x)\} + \frac{\delta}{2} < \infty. \quad (30)$$

From the definition of I there exists $\tilde{v} \in \mathcal{H}_H$ such that

$$\frac{1}{2} \|\tilde{v}\|_{\mathcal{H}_H}^2 \leq I(x_0) + \frac{\delta}{2}, \text{ and } x_0 = \mathcal{G}^0(\tilde{v}).$$

Applying Proposition 2 to the function $h \circ \mathcal{G}^\varepsilon$, one has

$$\begin{aligned} -\varepsilon \log \mathbb{E} \left(\exp \left\{ -\frac{h(X^\varepsilon)}{\varepsilon} \right\} \right) &= -\varepsilon \log \mathbb{E} \left(\exp \left\{ -\frac{h \circ \mathcal{G}^\varepsilon(\sqrt{\varepsilon}\omega)}{\varepsilon} \right\} \right) \\ &= \inf_{v \in \mathcal{A}_b} \mathbb{E} \left(h \circ \mathcal{G}^\varepsilon(\sqrt{\varepsilon}\omega + v) + \frac{1}{2} \|v\|_{\mathcal{H}_H}^2 \right). \end{aligned} \quad (31)$$

Thus, we have

$$\begin{aligned} \limsup_{\varepsilon \rightarrow 0} -\varepsilon \log \mathbb{E} \left(\exp \left\{ -\frac{h(X^\varepsilon)}{\varepsilon} \right\} \right) &= \limsup_{\varepsilon \rightarrow 0} \inf_{v \in \mathcal{A}_b} \mathbb{E} \left(h \circ \mathcal{G}^\varepsilon(\sqrt{\varepsilon}\omega + v) + \frac{1}{2} \|v\|_{\mathcal{H}_H}^2 \right) \\ &\leq \limsup_{\varepsilon \rightarrow 0} \mathbb{E} \left(h \circ \mathcal{G}^\varepsilon(\sqrt{\varepsilon}\omega + \tilde{v}) + \frac{1}{2} \|\tilde{v}\|_{\mathcal{H}_H}^2 \right) \\ &\leq \limsup_{\varepsilon \rightarrow 0} \mathbb{E} (h \circ \mathcal{G}^\varepsilon(\sqrt{\varepsilon}\omega + \tilde{v})) + I(x_0) + \frac{\delta}{2}. \end{aligned} \quad (32)$$

Since h is bounded and continuous, from (i) in Assumption 1, as ε go to 0 the last term in the above inequality equals

$$h \circ \mathcal{G}^0(\tilde{v}) + I(x_0) + \frac{\delta}{2} = h(x_0) + I(x_0) + \frac{\delta}{2}. \quad (33)$$

Combining (30), (32) and (33), we obtain

$$\limsup_{\varepsilon \rightarrow 0} -\varepsilon \log \mathbb{E} \left(\exp \left\{ -\frac{h(X^\varepsilon)}{\varepsilon} \right\} \right) \leq \inf_{x \in \mathcal{E}} \{h(x) + I(x)\} + \delta.$$

Since δ is arbitrary, the upper bound holds.

Proof of the lower bound (29): Fix $\delta > 0$. Then for each ε there exist $v^\varepsilon \in \mathcal{A}_b$ such that

$$\inf_{v \in \mathcal{A}_b} \mathbb{E} \left(h \circ \mathcal{G}^\varepsilon (\sqrt{\varepsilon} \omega + v) + \frac{1}{2} \|v\|_{\mathcal{H}_H}^2 \right) \geq \mathbb{E} \left(h \circ \mathcal{G}^\varepsilon (\sqrt{\varepsilon} \omega + v^\varepsilon) + \frac{1}{2} \|v^\varepsilon\|_{\mathcal{H}_H}^2 \right) - \delta. \quad (34)$$

Next note that from (31) and (34), for all ε , $\mathbb{E} \left(\frac{1}{2} \|v^\varepsilon\|_{\mathcal{H}_H}^2 \right) \leq 2M + \delta$, where $M = \|h\|_\infty$. Now define stopping times $\tau_N^\varepsilon = \inf\{t \in [0, 1] : \frac{1}{2} \int_0^t |\dot{v}^\varepsilon(s)|^2 ds \geq N\} \wedge 1$, where $v^\varepsilon = K_H \dot{v}^\varepsilon$. Denote $\dot{v}^{\varepsilon, N} = \dot{v}^\varepsilon 1_{[0, \tau_N^\varepsilon]}(s)$ and $v^{\varepsilon, N} = K_H \dot{v}^{\varepsilon, N}$. Then the processes $v^{\varepsilon, N}$ are in \mathcal{A}_b with $\frac{1}{2} \|v^{\varepsilon, N}\|_{\mathcal{H}_H}^2 \leq N$ a.s., and moreover

$$\mathbb{P}(v^\varepsilon \neq v^{\varepsilon, N}) \leq \mathbb{P} \left(\frac{1}{2} \int_0^1 |\dot{v}^\varepsilon(s)|^2 ds \geq N \right) = \mathbb{P} \left(\frac{1}{2} \|v^\varepsilon\|_{\mathcal{H}_H}^2 \geq N \right) \leq \frac{2M + \delta}{N}.$$

Choosing N large enough so that $\frac{2M(2M+\delta)}{N} \leq \delta$, we see that (34) holds with v^ε replaced with $v^{\varepsilon, N}$ and δ with 2δ . Henceforth, we will suppress N and denote $v^{\varepsilon, N}$ as v^ε . Note that, by definition,

$$\sup_{\varepsilon > 0} \frac{1}{2} \|v^\varepsilon\|_{\mathcal{H}_H}^2 \leq N \text{ a.s.} \quad (35)$$

Since $\delta > 0$ is arbitrary, in order to prove the lower bound (29), we now only need to show that

$$\liminf_{\varepsilon \rightarrow 0} \mathbb{E} \left(h \circ \mathcal{G}^\varepsilon (\sqrt{\varepsilon} \omega + v^\varepsilon) + \frac{1}{2} \|v^\varepsilon\|_{\mathcal{H}_H}^2 \right) \geq \inf_{x \in \mathcal{E}} \{h(x) + I(x)\}. \quad (36)$$

Choose a subsequence (still relabelled by ε) along which v^ε converges in distribution to v as S_N -valued random elements. Since h is a bounded and continuous function and the function on \mathcal{H}_H defined by $v \mapsto \frac{1}{2} \|v\|_{\mathcal{H}_H}^2$ is lower semi-continuous with respect to the weak topology, using (i) in Assumption 1 and Fatou's lemma, we obtain

$$\begin{aligned} \liminf_{\varepsilon \rightarrow 0} \mathbb{E} \left(h \circ \mathcal{G}^\varepsilon (\sqrt{\varepsilon} \omega + v^\varepsilon) + \frac{1}{2} \|v^\varepsilon\|_{\mathcal{H}_H}^2 \right) &\geq \mathbb{E} \left(h \circ \mathcal{G}^0(v) + \frac{1}{2} \|v\|_{\mathcal{H}_H}^2 \right) \\ &\geq \inf_{\{(x, v) : x = \mathcal{G}^0(v)\}} \left\{ h(x) + \frac{1}{2} \|v\|_{\mathcal{H}_H}^2 \right\} \\ &\geq \inf_{x \in \mathcal{E}} \{h(x) + I(x)\}. \end{aligned}$$

This completes the proof of the lower bound.

Compactness of Level Sets: In order to prove that I is a rate function, we need to show that, for any $0 < M < \infty$, the level set $\{x : I(x) \leq M\}$ is compact. In order to prove this, we will show that

$$\{x : I(x) \leq M\} = \bigcap_{n=1}^{\infty} \Gamma_{M+\frac{1}{n}}.$$

The compactness of the level set $\{x : I(x) \leq M\}$ will then follow, since from (ii) in Assumption 1 the set $\Gamma_{M+\frac{1}{n}}$ is compact for each n .

Let $x \in \mathcal{E}$ with $I(x) \leq M$. Then, for each n , from the definition of $I(x)$ there exists $v^n \in \mathcal{H}_H$ such that $\frac{1}{2}\|v^n\|_{\mathcal{H}_H}^2 \leq M + \frac{1}{n}$ and $x = \mathcal{G}^0(v^n)$. This shows $x \in \bigcap_{n=1}^{\infty} \Gamma_{M+\frac{1}{n}}$. Conversely, suppose $x \in \bigcap_{n=1}^{\infty} \Gamma_{M+\frac{1}{n}}$. Then, for each n , there exists $v^n \in S_{M+\frac{1}{n}}$ such that $x = \mathcal{G}^0(v^n)$. Thus, we have $I(x) \leq \frac{1}{2}\|v^n\|_{\mathcal{H}_H}^2 \leq M + \frac{1}{n}$, for all n . By letting n go to infinity, we conclude that $I(x) \leq M$. This completes the proof that I is a rate function. \square

4.3 Proof of Proposition 3(b)

Proof Without loss of generality we assume that $d = m = 1$. Now, fix $0 \leq s < t \leq 1$. In the proof, we will use C to denote a generic positive constant which may depend on α and all the constants appearing in Assumption 2 but is independent of s and t . This generic constant may vary from line to line.

From (19), we have

$$|x_t - x_s| \leq \int_s^t |b(r, x_r)| dr + \left| \int_s^t \sigma(r, x_r) dg_r \right|. \quad (37)$$

Using the Lipschitz property and linear growth of b we see that

$$\begin{aligned} \int_s^t |b(r, x_r)| dr &\leq L \int_s^t (1 + |x_s| + \|x\|_{s,t,1-\alpha}(r-s)^{1-\alpha}) dr \\ &\leq C(1 + |x_s|)(t-s) + C\|x\|_{s,t,1-\alpha}(t-s)^{2-\alpha}. \end{aligned} \quad (38)$$

Recalling from Remark 3 that $s \mapsto \sigma(s, x(s)) \in C^\delta([0, 1] : \mathbb{R}^m)$ for some $\delta > 1 - H$ and using the fractional integration by parts formula given in Proposition 1, we obtain

$$\left| \int_s^t \sigma(r, x_r) dg_r \right| \leq \int_s^t |D_{s+}^\alpha \sigma(r, x_r) D_{t-}^{1-\alpha} g_{t-}(r)| dr.$$

From (4) and the conditions on σ in Assumption 2(i) we see that, for $s \leq r \leq t$

$$\begin{aligned}
|D_{s+}^\alpha \sigma(r, x_r)| &\leq C \left(\frac{K(1 + |x_s|) + M|x_r - x_s|}{(r-s)^\alpha} + \alpha \int_s^r \frac{M(r-u)^\lambda + M|x_r - x_u|}{(r-u)^{\alpha+1}} du \right) \\
&\leq C(1 + |x_s|)(r-s)^{-\alpha} + C(r-s)^{\lambda-\alpha} + C\|x\|_{s,t,1-\alpha}(r-s)^{1-2\alpha}.
\end{aligned}$$

It is easy to verify that $|D_t^{1-\alpha} g_{t-}(r)| \leq C\|g\|_{1-\alpha+\delta}(t-r)^\delta$. Thus,

$$\begin{aligned}
\left| \int_s^t \sigma(r, x_r) dg_r \right| &\leq C\|g\|_{1-\alpha+\delta}(1 + |x_s|)(t-s)^{1-\alpha+\delta} + C\|g\|_{1-\alpha+\delta}(t-s)^{1+\lambda-\alpha+\delta} \\
&\quad + C\|g\|_{1-\alpha+\delta}\|x\|_{s,t,1-\alpha}(t-s)^{2-2\alpha+\delta} \\
&\leq C\|g\|_{1-\alpha+\delta}(1 + |x_s|)(t-s)^{1-\alpha+\delta} \\
&\quad + C\|g\|_{1-\alpha+\delta}\|x\|_{s,t,1-\alpha}(t-s)^{2-2\alpha+\delta}.
\end{aligned} \tag{39}$$

$$(40)$$

Hence, from (37)-(39) we get

$$\|x\|_{s,t,1-\alpha} \leq C_0(1 + \|g\|_{1-\alpha+\delta})(1 + |x_s|) + C_0(1 + \|g\|_{1-\alpha+\delta})\|x\|_{s,t,1-\alpha}(t-s)^{1-\alpha} \tag{41}$$

for some $C_0 > 0$ independent of s, t . Choose

$$\Delta = \left(\frac{1}{2C_0(1 + \|g\|_{1-\alpha+\delta})} \right)^{\frac{1}{1-\alpha}}.$$

Then, for all s, t with $t-s \leq \Delta$, we have

$$\|x\|_{s,t,1-\alpha} \leq 2C_0(1 + \|g\|_{1-\alpha+\delta})(1 + |x_s|). \tag{42}$$

Therefore, for s, t as above,

$$\|x\|_{s,t,\infty} \leq |x_s| + \|x\|_{s,t,1-\alpha}(t-s)^{1-\alpha} \leq |x_s| + 2C_0(1 + \|g\|_{1-\alpha+\delta})(1 + |x_s|)(t-s)^{1-\alpha}.$$

In particular, for all s, t with $t-s \leq \Delta$, we have $\|x\|_{s,t,\infty} \leq 2|x_s| + 1$, and hence

$$\sup_{0 \leq r \leq t} |x_r| \leq 2 \sup_{0 \leq r \leq s} |x_r| + 1. \tag{43}$$

Now divide the interval $[0, 1]$ into $n = \left\lceil \frac{1}{\Delta} \right\rceil + 1$ subintervals with equal lengths, and denote $t_k = \frac{k}{n}$, $k = 0, 1, \dots, n$. Applying (43) in each subinterval we obtain

$$\sup_{0 \leq r \leq 1} |x_r| \leq 2^n |x_0| + 2^{n-1} + 2^{n-2} + \dots + 1 \leq 2^n(|x_0| + 1).$$

Thus, recalling the definition of n, Δ ,

$$\|x\|_\infty \leq 2^{1+(2C_0(1+\|g\|_{1-\alpha+\delta}))\frac{1}{1-\alpha}} (|x_0| + 1). \tag{44}$$

Now, fix u, v in $[0, 1]$ with $u < v$. If $v-u \leq \Delta$ then (42) and (44) yield

$$\begin{aligned}
\frac{|x_v - x_u|}{(v - u)^{1-\alpha}} &\leq 2C_0(1 + \|g\|_{1-\alpha+\delta})(1 + \|x\|_\infty) \\
&\leq 2C_0(1 + \|g\|_{1-\alpha+\delta}) \left(1 + 2^{1+(2C_0(1+\|g\|_{1-\alpha+\delta}))^{\frac{1}{1-\alpha}}} (|x_0| + 1) \right). \quad (45)
\end{aligned}$$

If $v - u > \Delta$, there are $k_0 = k_0(u, v)$, $k_1 = k_1(u, v) \in \{1, \dots, n\}$ such that $k_0 \leq k_1$, $t_{k_0} - u \leq 1/n$, $v - t_{k_1} \leq 1/n$, and

$$u < t_k < v, \text{ for all } k = k_0, k_0 + 1, \dots, k_1.$$

Then, applying (42) and (44) we obtain

$$\begin{aligned}
\frac{|x_v - x_u|}{(v - u)^{1-\alpha}} &\leq \frac{|x_v - x_{t_{k_1}}|}{(v - t_{k_1})^{1-\alpha}} + \frac{|x_{t_{k_0}} - x_u|}{(t_{k_0} - u)^{1-\alpha}} + \sum_{k=k_0+1}^{k_1} \frac{|x_{t_k} - x_{t_{k-1}}|}{(t_k - t_{k-1})^{1-\alpha}} \\
&\leq 2nC_0(1 + \|g\|_{1-\alpha+\delta})(1 + \|x\|_\infty) \\
&\leq 2C_0(1 + \|g\|_{1-\alpha+\delta}) \left((2C_0(1 + \|g\|_{1-\alpha+\delta}))^{\frac{1}{1-\alpha}} + 2 \right) \\
&\quad \times \left(1 + 2^{1+(2C_0(1+\|g\|_{1-\alpha+\delta}))^{\frac{1}{1-\alpha}}} (|x_0| + 1) \right). \quad (46)
\end{aligned}$$

The estimates in (44)-(46) imply (20) and (21) completing the proof. \square

4.4 Proof of Theorem 2

Throughout this subsection we assume that Assumption 2 holds. The following lemma is immediate from the strong existence and pathwise uniqueness result in Proposition 4.

Lemma 5 *For each $\varepsilon \in (0, 1)$, there exists a measurable function*

$$\mathcal{G}^\varepsilon : C_0([0, 1] : \mathbb{R}^d) \rightarrow C([0, 1] : \mathbb{R}^m)$$

such that, for any probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with a fractional Brownian motion B^H defined on it,

$$X^\varepsilon = \mathcal{G}^\varepsilon(\sqrt{\varepsilon}B^H) \quad (47)$$

is the unique pathwise solution of (17). Moreover, for any $\alpha \in (1-H, \min\{\frac{1}{2}, \lambda, \frac{\gamma}{1+\gamma}\})$, the following estimate holds:

$$\sup_{\varepsilon \in (0, 1)} \mathbb{E} \|X^\varepsilon\|_{1-\alpha}^p < \infty, \text{ for all } p \geq 1. \quad (48)$$

Remark 4 The estimate (48) implies the tightness of $\{X^\varepsilon\}$ in $C([0, 1] : \mathbb{R}^m)$. Moreover, by a similar argument as in the proof of Lemma 7, it can be seen that the sequence of processes $\{X^\varepsilon\}$ converges in distribution to the solution ϕ to the fol-

lowing ordinary differential equation

$$\phi_t = x_0 + \int_0^t b(s, \phi_s) ds. \quad (49)$$

The existence and uniqueness of the solution to (49) follows from the Lipschitz condition on b in Assumption 2.

The following Girsanov theorem is a multi-dimensional version of [11, Theorem 4.9]. Recall the operator K_H from (11) and the canonical space $(\Omega, \mathcal{F}, \{\mathcal{F}_t^H\}, \mathbb{P})$ along with the canonical process B^H on this space from Section 4.1.

Theorem 3 *Let $u = K_H \dot{u} \in \mathcal{A}$ be such that*

$$\mathbb{E} \left(\exp \left\{ - \int_0^1 \dot{u}(s) \cdot dB_s - \frac{1}{2} \int_0^1 |\dot{u}(s)|^2 ds \right\} \right) = 1,$$

where B is the d -dimensional standard Brownian motion defined by (27) on the probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t^H\}, \mathbb{P})$ and

$$\int_0^1 \dot{u}_s \cdot dB_s = \sum_{i=1}^d \int_0^1 \dot{u}_s^i dB_s^i$$

is the usual Itô integral. Let $\tilde{\mathbb{P}}$ be the probability measure on (Ω, \mathcal{F}) defined by

$$\frac{d\tilde{\mathbb{P}}}{d\mathbb{P}} = \exp \left\{ - \int_0^1 \dot{u}(s) \cdot dB_s - \frac{1}{2} \int_0^1 |\dot{u}(s)|^2 ds \right\}.$$

Then the law of the process

$$\tilde{B}^H = \left\{ \tilde{B}_t^H = B_t^H + u_t = B_t^H + \int_0^t K_H(t, s) \dot{u}_s ds : t \in [0, 1] \right\} \quad (50)$$

under the probability $\tilde{\mathbb{P}}$ is the same as the law of the process B^H under the probability \mathbb{P} . Namely, the process \tilde{B}^H is a fractional Brownian motion under $\tilde{\mathbb{P}}$.

Let $v \in \mathcal{A}_b$ and consider the controlled version of the SDE (17) given as

$$X_t^{\varepsilon, v} = x_0 + \int_0^t b(s, X_s^{\varepsilon, v}) ds + \int_0^t \sigma(s, X_s^{\varepsilon, v}) dv_s + \sqrt{\varepsilon} \int_0^t \sigma(s, X_s^{\varepsilon, v}) dB_s^H. \quad (51)$$

The following result gives the wellposedness of the above equation.

Lemma 6 *Let \mathcal{G}^ε be as in Lemma 5 and let $v \in \mathcal{A}_b$. Define*

$$X^{\varepsilon, v} = \mathcal{G}^\varepsilon(\sqrt{\varepsilon} B^H + v).$$

Then $X^{\varepsilon, v}$ is the unique pathwise solution of (51). Moreover, for any fixed α with $1 - H < \alpha < \min\{\frac{1}{2}, \lambda, \frac{\gamma}{1+\gamma}\}$ and for any $0 < \delta < \alpha - (1 - H)$, we have, a.s.,

$$\begin{aligned} \|X^{\varepsilon,v}\|_{1-\alpha} &\leq \tilde{C}_1(1+|x_0|)(1+\|B^H\|_{1-\alpha+\delta}^\kappa + \|v\|_{\mathcal{H}_H}^\kappa)(1+\|B^H\|_{1-\alpha+\delta} + \|v\|_{\mathcal{H}_H}) \\ &\quad \times (1 + \exp\{\tilde{C}_2(\|B^H\|_{1-\alpha+\delta}^\kappa + \|v\|_{\mathcal{H}_H}^\kappa)\}), \end{aligned} \quad (52)$$

for all ε , where $\kappa = \frac{1}{1-\alpha}$ and the constants \tilde{C}_1 and \tilde{C}_2 depend only on α , δ and all the constants appearing in Assumption 2.

Proof Fix $v = K_H \dot{v} \in \mathcal{A}_b$. Note that

$$\mathbb{E} \left(\exp \left\{ -\frac{1}{\sqrt{\varepsilon}} \int_0^1 \dot{v}_s \cdot dB_s - \frac{1}{2\varepsilon} \int_0^1 |\dot{v}_s|^2 ds \right\} \right) = 1,$$

where B is the d -dimensional standard Brownian motion defined by (27). Let $\tilde{\mathbb{P}}$ be the probability measure defined by

$$\frac{d\tilde{\mathbb{P}}}{d\mathbb{P}} = \exp \left\{ -\frac{1}{\sqrt{\varepsilon}} \int_0^1 \dot{v}_s \cdot dB_s - \frac{1}{2\varepsilon} \int_0^1 |\dot{v}_s|^2 ds \right\}.$$

Then, by Theorem 3, the process $\tilde{B}^H = B^H + \frac{1}{\sqrt{\varepsilon}}v$ is a fractional Brownian motion on $(\Omega, \mathcal{F}, \tilde{\mathbb{P}}, \{\mathcal{F}_t\})$. By Lemma 5, the process $X^{\varepsilon,v} = \mathcal{G}^\varepsilon(\sqrt{\varepsilon}B^H + v) = \mathcal{G}^\varepsilon(\sqrt{\varepsilon}\tilde{B}^H)$ is the unique solution to (17) on $(\Omega, \mathcal{F}, \tilde{\mathbb{P}}, \{\mathcal{F}_t\})$. Note that the equation (17) with \tilde{B}^H is precisely the same as the equation (51). Since the probability measures $\tilde{\mathbb{P}}$ and \mathbb{P} are equivalent (i.e. mutually absolutely continuous), we see that $X^{\varepsilon,v}$ is a strong solution of (51) on $(\Omega, \mathcal{F}, \mathbb{P}, \{\mathcal{F}_t\})$. Uniqueness of solutions of (51) is argued similarly using the uniqueness result in Lemma 5 and Girsanov's theorem (Theorem 3) once more.

Now fix α with $1-H < \alpha < \min\{\frac{1}{2}, \lambda, \frac{\gamma}{1+\gamma}\}$ and $0 < \delta < \alpha - (1-H)$. From Lemma 2, it follows that

$$\|v\|_{1-\alpha+\delta} \leq \|v\|_H \leq \|v\|_{\mathcal{H}_H}.$$

Then, applying Proposition 3 (with $g_s = \sqrt{\varepsilon}B_s^H + v_s$), we obtain (52) for some constants \tilde{C}_1 and \tilde{C}_2 , depending only on the constants C_1, C_2 in Proposition 3. \square

Next, for $v \in \mathcal{H}_H$, consider the following deterministic equation

$$X^{0,v} = x_0 + \int_0^t b(s, X_s^{0,v})ds + \int_0^t \sigma(s, X_s^{0,v})dv_s. \quad (53)$$

Due to Assumption 2 and the Hölder continuity of v proved in Lemma 2, Proposition 3 implies the existence and uniqueness of the solution to (53). Furthermore, for any $\alpha \in (1-H, \min\{\frac{1}{2}, \lambda, \frac{\gamma}{1+\gamma}\})$, from (20) and the Hölder continuity of v it follows that

$$\|X^{0,v}\|_{1-\alpha} \leq C_3(1+|x_0|)(1+\|v\|_{\mathcal{H}_H}^\kappa)(1+\|v\|_{\mathcal{H}_H})(1+\exp\{C_2\|v\|_{\mathcal{H}_H}^\kappa\}), \quad (54)$$

where $\kappa = \frac{1}{1-\alpha}$ and C_2, C_3 are the constants in Proposition 3.

Now, define the map $\mathcal{G}^0 : \mathcal{H}_H \rightarrow C([0, 1] : \mathbb{R}^m)$ by

$$\mathcal{G}^0(v) = X^{0,v}, \quad v \in \mathcal{H}_H, \quad (55)$$

where $X^{0,v}$ is the unique solution to (53).

Lemma 7 *Let the map \mathcal{G}^0 be as in (55). Then for each $N < \infty$, the restriction of the map \mathcal{G}^0 to S_N is continuous.*

Proof Let $v^n = K_H(\dot{v}^n)$ and $v = K_H\dot{v}$ be in S_N , and assume that v^n converges to v in S_N (i.e. under the weak topology on \mathcal{H}_H).

Let X^{0,v^n} be the solution to (53) with v replaced by v^n , that is, it satisfies the following equation

$$X_t^{0,v^n} = x_0 + \int_0^t b(s, X_s^{0,v^n}) ds + \int_0^t \sigma(s, X_s^{0,v^n}) dv_s^n. \quad (56)$$

Now, fix $\alpha \in (1 - H, \min\{\frac{1}{2}, \lambda, \frac{\gamma}{1+\gamma}\})$. Note that $\sup_n \|v^n\|_{\mathcal{H}_H} \leq \sqrt{2N} < \infty$, and hence, by (54), we see that $\sup_n \|X^{0,v^n}\|_{1-\alpha} < \infty$. Thus, the sequence $\{X^{0,v^n}\}$ is relatively compact in $C([0, 1] : \mathbb{R}^m)$ and therefore for any subsequence of $\{X^{0,v^n}\}$, there is a further subsequence (still relabeled by n) such that X^{0,v^n} converges to X in $C([0, 1] : \mathbb{R}^m)$.

For any $h = K_H\dot{h} \in \mathcal{H}_H$, (13) implies that h is differentiable and

$$h'(t) = c_H t^{H-\frac{1}{2}} (I_{0+}^{H-\frac{1}{2}}(\psi^{-1}\dot{h}))(t) = \frac{c_H t^{H-\frac{1}{2}}}{\Gamma(H-\frac{1}{2})} \int_0^t (t-s)^{H-\frac{3}{2}} s^{\frac{1}{2}-H} \dot{h}(s) ds, \quad (57)$$

where ψ is as in Lemma 1. Thus, if $f \in C([0, 1] : \mathbb{R}^d)$, the Riemann-Stieltjes integral $\int_0^1 f(s) dh(s)$ is well-defined and equals $\int_0^1 f(s) h'(s) ds$.

From the Lipschitz condition on σ and changing the order of integration, we have

$$\begin{aligned} & \left| \int_0^t \sigma(s, X_s^{0,v^n}) dv_s^n - \int_0^t \sigma(s, X_s) dv_s \right| \\ & \leq \left| \int_0^t \sigma(s, X_s^{0,v^n}) dv_s^n - \int_0^t \sigma(s, X_s) dv_s^n \right| + \left| \int_0^t \sigma(s, X_s) dv_s^n - \int_0^t \sigma(s, X_s) dv_s \right| \\ & = \frac{c_H}{\Gamma(H-\frac{1}{2})} \left| \int_0^t (\sigma(s, X_s^{0,v^n}) - \sigma(s, X_s)) s^{H-\frac{1}{2}} \left(\int_0^s (s-u)^{H-\frac{3}{2}} u^{\frac{1}{2}-H} \dot{v}_u^n du \right) ds \right| \\ & \quad + \frac{c_H}{\Gamma(H-\frac{1}{2})} \left| \int_0^t \sigma(s, X_s) s^{H-\frac{1}{2}} \left(\int_0^s (s-u)^{H-\frac{3}{2}} u^{\frac{1}{2}-H} (\dot{v}_u^n - \dot{v}_u) du \right) ds \right| \\ & \leq \frac{Mc_H}{\Gamma(H-\frac{1}{2})} \|X^{0,v^n} - X\|_\infty \int_0^t u^{\frac{1}{2}-H} |\dot{v}_u^n| \left(\int_u^t s^{H-\frac{1}{2}} (s-u)^{H-\frac{3}{2}} ds \right) du \\ & \quad + \frac{c_H}{\Gamma(H-\frac{1}{2})} \left| \int_0^t u^{\frac{1}{2}-H} \left(\int_u^t s^{H-\frac{1}{2}} (s-u)^{H-\frac{3}{2}} \sigma(s, X_s) ds \right) (\dot{v}_u^n - \dot{v}_u) du \right|. \quad (58) \end{aligned}$$

Observe that the first term on the right-hand side of (58) is no more than

$$\begin{aligned}
& C \|X^{0,v^n} - X\|_\infty \int_0^1 u^{\frac{1}{2}-H} |\dot{v}_u^n| \left(\int_u^1 (s-u)^{H-\frac{3}{2}} ds \right) du \\
& \leq C \|X^{0,v^n} - X\|_\infty \int_0^1 u^{\frac{1}{2}-H} |\dot{v}_u^n| du \\
& \leq C \|X^{0,v^n} - X\|_\infty \|\dot{v}^n\|_{L^2} \\
& = C \|X^{0,v^n} - X\|_\infty \|v^n\|_{\mathcal{H}_H} \leq C \sqrt{2N} \|X^{0,v^n} - X\|_\infty,
\end{aligned} \tag{59}$$

where C is a constant depending on M and H which may vary from line to line. From the uniform convergence of X^{0,v^n} to X the right side of (59) converges to 0 and thus we obtain

$$\lim_{n \rightarrow \infty} \|X^{0,v^n} - X\|_\infty \int_0^t u^{\frac{1}{2}-H} |\dot{v}_u^n| \left(\int_u^t s^{H-\frac{1}{2}} (s-u)^{H-\frac{3}{2}} ds \right) du = 0. \tag{60}$$

For the second term on the right-hand side of (58), we first study the function (in u) $f^t(u) = 1_{[0,t]}(u) \int_u^t s^{H-\frac{1}{2}} (s-u)^{H-\frac{3}{2}} \sigma(s, X_s) ds$. From (18), we get

$$|f^t(u)| \leq K(1 + \|X\|_\infty) 1_{[0,t]}(u) \int_u^t s^{H-\frac{1}{2}} (s-u)^{H-\frac{3}{2}} ds \leq C(1 + \|X\|_\infty), \tag{61}$$

where C is a constant depending on K and H . Thus it follows that the function $u \mapsto u^{\frac{1}{2}-H} f^t(u)$ is in $L^2([0, 1] : \mathbb{R}^{m \times d})$. Thus, from the convergence of v^n to v (in the weak topology), we have, for every $t \in [0, 1]$,

$$\lim_{n \rightarrow \infty} \left| \int_0^t u^{\frac{1}{2}-H} \left(\int_u^t s^{H-\frac{1}{2}} (s-u)^{H-\frac{3}{2}} \sigma(s, X_s) ds \right) (\dot{v}_u^n - \dot{v}_u) du \right| = 0. \tag{62}$$

From (58), (60), (62), we see that, for each $t \in [0, 1]$,

$$\lim_{n \rightarrow \infty} \int_0^t \sigma(s, X_s^{0,v^n}) dv_s^n = \int_0^t \sigma(s, X_s) dv_s. \tag{63}$$

From (56), (63), the uniform convergence of X^{0,v^n} to X and the Lipschitz condition on b , we have by sending n to infinity that X satisfies equation (53). Recalling that (53) has a unique solution, we get $X = X^{0,v}$. Thus, by a standard subsequential argument, we see that the full sequence $\mathcal{G}^0(v^n) = X^{0,v^n}$ converges to $\mathcal{G}^0(v) = X^{0,v}$ in $C([0, 1] : \mathbb{R}^m)$. The proof is complete. \square

Lemma 8 Fix $N \in (0, \infty)$. Let $\{v^\varepsilon\} \subset \mathcal{A}_b$ be a family of S_N -valued random variables on $(\Omega, \mathcal{F}, \mathbb{P})$ such that v^ε converges in distribution to v in the weak topology on S_N . Then $\mathcal{G}^\varepsilon(\sqrt{\varepsilon}\omega + v^\varepsilon)$ converges to $\mathcal{G}^0(v)$ in distribution.

Proof Note from Lemma 6 that $\mathcal{G}^\varepsilon(\sqrt{\varepsilon}\omega + v^\varepsilon) = X^{\varepsilon, v^\varepsilon}$, where $X^{\varepsilon, v^\varepsilon}$ is the unique solution to the following SDE:

$$X_t^{\varepsilon, v^\varepsilon} = x_0 + \int_0^t b(s, X_s^{\varepsilon, v^\varepsilon}) ds + \int_0^t \sigma(s, X_s^{\varepsilon, v^\varepsilon}) dv_s + \sqrt{\varepsilon} \int_0^t \sigma(s, X_s^{\varepsilon, v^\varepsilon}) dB_s^H. \quad (64)$$

For any fixed α with $1 - H < \alpha < \min\{\frac{1}{2}, \lambda, \frac{\gamma}{1+\gamma}\}$, from (52) and Fernique's theorem (cf. [16]) we have

$$\sup_{\varepsilon \in (0,1)} \mathbb{E} \|X^{\varepsilon, v^\varepsilon}\|_{1-\alpha}^p < \infty, \text{ for all } p \geq 1. \quad (65)$$

In particular $\{X^{\varepsilon, v^\varepsilon}\}$ is tight in $C([0, 1] : \mathbb{R}^m)$. This tightness together with the compactness of S_N under topology of the weak convergence yield that for any subsequence of $\{(X^{\varepsilon, v^\varepsilon}, v^\varepsilon)\}$ there is a further subsequence (still relabeled by ε) such that $(X^{\varepsilon, v^\varepsilon}, v^\varepsilon)$ converges weakly to (X, v) in $C([0, 1] : \mathbb{R}^m) \times S_N$.

Define the mapping $F^{b, \sigma} : C([0, 1] : \mathbb{R}^m) \times S_N \rightarrow C([0, 1] : \mathbb{R}^m)$ as

$$F_t^{b, \sigma}(x, u) = x_0 + \int_0^t b(s, x_s) ds + \int_0^t \sigma(s, x_s) du_s,$$

for $t \in [0, 1]$ and $(x, u) \in C([0, 1] : \mathbb{R}^m) \times S_N$. It is easy to see that the right side defines a function in $C([0, 1] : \mathbb{R}^m)$ for any $(x, u) \in C([0, 1] : \mathbb{R}^m) \times S_N$. We will now show that $F^{b, \sigma}$ is a continuous mapping from $C([0, 1] : \mathbb{R}^m) \times S_N$ to $C([0, 1] : \mathbb{R}^m)$. Let (x^n, u^n) converge to (x, u) in $C([0, 1] : \mathbb{R}^m) \times S_N$, and denote $u^n = K_H \dot{u}^n$ and $u = K_H \dot{u}$ with $u^n, u \in L^2([0, 1] : \mathbb{R}^d)$. By the Lipschitz condition on b and analogous arguments as in (58), (60) and (62), we obtain

$$\begin{aligned} & \sup_{0 \leq t \leq 1} |F_t^{b, \sigma}(x^n, u^n) - F_t^{b, \sigma}(x, u)| \\ & \leq L \|x^n - x\|_\infty + \frac{Mc_H}{\Gamma(H - \frac{1}{2})} \|x^n - x\|_\infty \int_0^1 r^{\frac{1}{2}-H} |\dot{u}_r^n| \left(\int_r^1 s^{H-\frac{1}{2}} (s-r)^{H-\frac{3}{2}} ds \right) dr \\ & \quad + \frac{c_H}{\Gamma(H - \frac{1}{2})} \left| \int_0^1 r^{\frac{1}{2}-H} \left(\int_r^1 s^{H-\frac{1}{2}} (s-r)^{H-\frac{3}{2}} \sigma(s, x_s) ds \right) (\dot{u}_r^n - \dot{u}_r) dr \right| \rightarrow 0, \end{aligned} \quad (66)$$

as $n \rightarrow \infty$, which proves the continuity of $F^{b, \sigma}$. Thus, by the continuous mapping theorem, $\{F^{b, \sigma}(X^{\varepsilon, v^\varepsilon}, v^\varepsilon) = x_0 + \int_0^\cdot b(s, X_s^{\varepsilon, v^\varepsilon}) ds + \int_0^\cdot \sigma(s, X_s^{\varepsilon, v^\varepsilon}) dv_s^\varepsilon\}$ converges in distribution to $\{F^{b, \sigma}(X, v) = x_0 + \int_0^\cdot b(s, X_s) ds + \int_0^\cdot \sigma(s, X_s) dv_s\}$ in $C([0, 1] : \mathbb{R}^m)$.

Now for any $0 < \delta < \alpha - (1 - H)$, from the observation in (39) with (x, g) replaced by $(X^{\varepsilon, v^\varepsilon}, B^H)$, we obtain

$$\begin{aligned} \left\| \int_0^\cdot \sigma(s, X_s^{\varepsilon, v^\varepsilon}) dB_s^H \right\|_{1-\alpha} & \leq C(1 + \|X^{\varepsilon, v^\varepsilon}\|_\infty + \|X^{\varepsilon, v^\varepsilon}\|_{1-\alpha}) \|B^H\|_{1-\alpha+\delta} \\ & \leq C(1 + \|X^{\varepsilon, v^\varepsilon}\|_{1-\alpha}) \|B^H\|_{1-\alpha+\delta}. \end{aligned}$$

Then, from (65) we get

$$\sup_{\varepsilon \in (0,1)} \mathbb{E} \left\| \int_0^\cdot \sigma(s, X_s^{\varepsilon, v^\varepsilon}) dB_s^H \right\|_{1-\alpha}^p < \infty, \text{ for all } p \geq 1,$$

which implies that $\{\sqrt{\varepsilon} \int_0^\cdot \sigma(s, X_s^{\varepsilon, v^\varepsilon}) dB_s^H\}$ converges in probability to 0 in $C([0, 1] : \mathbb{R}^m)$.

Combining the above observations we now have that

$$x_0 + \int_0^\cdot b(s, X_s^{\varepsilon, v^\varepsilon}) ds + \int_0^\cdot \sigma(s, X_s^{\varepsilon, v^\varepsilon}) dv_s + \sqrt{\varepsilon} \int_0^\cdot \sigma(s, X_s^{\varepsilon, v^\varepsilon}) dB_s^H$$

converges in distribution in $C([0, 1] : \mathbb{R}^m)$ to

$$x_0 + \int_0^\cdot b(s, X_s) ds + \int_0^\cdot \sigma(s, X_s) dv_s.$$

Since $X^{\varepsilon, v^\varepsilon}$ converges to X in distribution in $C([0, 1] : \mathbb{R}^m)$ and that the solution to (53) is unique, we conclude that $X^{\varepsilon, v^\varepsilon} = \mathcal{G}^\varepsilon(\sqrt{\varepsilon}\omega + v^\varepsilon)$ converges to $X^{0, v} = \mathcal{G}^0(v)$ in distribution in $C([0, 1] : \mathbb{R}^m)$. \square

Proof of Theorem 2. Note that the rate function I defined in (22) coincides with the one in (16) for \mathcal{G}^0 defined in (55). From Theorem 1, to prove Theorem 2, it is sufficient to verify the conditions in Assumption 1 for \mathcal{G}^ε defined in Lemma 5 and \mathcal{G}^0 in (55). Lemma 8 shows that the Assumption 1(i) is satisfied. Also, since S_N is compact in the weak topology, Lemma 7 implies that the set

$$\Gamma_N = \{\mathcal{G}^0(v) : v \in S_N\}$$

is a compact subset of \mathcal{E} , which verifies Assumption 1(ii). \square

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